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On the neutral component of the Jacobian of a real algebraic curve having many components

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ABSTRACT

Let C be a real algebraic curve of genus $g \geq 1$ having at least g real components. We show that there is an embedding of C into $\mathbb{P}^{2g}$ as a curve of degree $3g$ which induces a group structure on a connected component X of the set of effective divisors on C of degree g. Moreover, after having chosen a base point $O \in X$, there is a natural isomorphism of X onto the neutral real component of the Jacobian of C. This furnishes an explicit description of the group structure on the neutral real component of the Jacobian of a real algebraic curve of genus $g \geq 1$ having many real components. If $g = 1$, one recovers the geometric description of the group structure on the neutral real component of a real elliptic curve.

1. INTRODUCTION

A nonsingular geometrically irreducible complete algebraic curve C of genus 1 over a field K having a rational point O admits an explicit geometric description of its Jacobian [12]. This description can be divided into two stages.

The first stage is to consider C embedded in $\mathbb{P}^2$ as a nonsingular cubic curve via the linear system $|3O|$. This embedding induces a group structure on the set $C(L)$ of L-rational points, for any field extension L of K. The reason behind the existence of this group structure is the fact that, for all $P, Q \in C(L)$, the divisor $3O - P - Q$ is nonspecial, by Riemann-Roch, and of degree 1. Moreover, if L' is a field extension of L then the group structures on $C(L)$ and $C(L')$ are compatible. Hence, C acquires the structure of an algebraic group over K.

The second stage is to remark that the map $\mathrm{cl}(L)\colon P \mapsto \mathcal{O}(P - O)$ from $C(L)$ into the group $\mathrm{Pic}^0_C(L)$ of L-rational points of the Jacobian Pic^0_C of C is a morphism of groups, for any extension L/K. By Riemann-Roch, the morphism $\mathrm{cl}(L)$ is an isomorphism. Again, if L'/L is a field extension, the isomorphisms $\mathrm{cl}(L)$ and $\mathrm{cl}(L')$ are compatible. Hence, one has an induced isomorphism $\mathrm{cl}\colon C \to \mathrm{Pic}_C^0$ of algebraic groups over K.

When one wants to generalize the above explicit geometric description of the Jacobian to higher genus g, both stages give rise to difficulties:

As for the second stage, one cannot do better than replacing C by the algebraic variety $\mathrm{Div}_C^{g,\mathrm{eff}}$ over K of effective divisors on C of degree g. Instead of the morphism $\mathrm{cl}\colon C \to \mathrm{Pic}_C^0$, one considers a morphism of algebraic varieties

$$\mathrm{cl}\colon \mathrm{Div}_C^{g,\mathrm{eff}} \to \mathrm{Pic}_C^0.$$

On L-rational points, $\mathrm{cl}(L)$ associates to a divisor D the isomorphism class of the line bundle $\mathcal{O}(D - O)$, for some fixed K-rational divisor O on C of degree g. But, the morphism cl is only a birational isomorphism. Its exceptional locus is exactly the locus of special effective divisors on C of degree g.

As for the first stage, one embeds C into $\mathbb{P}^{2g}$ as a curve of degree $3g$ via the linear system $|3O|$. In order to have an induced structure of an algebraic group on $\mathrm{Div}_C^{g,\mathrm{eff}}$ over K, the divisor $3O - P - Q$ has to be nonspecial, for all L-rational effective divisors P and Q on C of degree g and for all extensions L/K. However, there are, for any K-rational divisor O on C of degree g, an extension L/K and effective L-rational divisors P and Q of degree g such that $3O - P - Q$ is special. Therefore, one only gets a rational group law on $\mathrm{Div}_C^{g,\mathrm{eff}}$.

To summarize, if the genus g of C is at least 2, one has only a rational group law on the algebraic variety $\mathrm{Div}_C^{g,\mathrm{eff}}$ over K and only a birational isomorphism cl from $\mathrm{Div}_C^{g,\mathrm{eff}}$ onto Pic_C^0, compatible with the rational group laws.

Now, the object of this paper is to show that if C is a real algebraic curve having many real components then, for a good choice of the divisor O, there is a connected component X of $\mathrm{Div}_C^{g,\mathrm{eff}}(\mathbb{R})$ on which the rational group law is a true group law. And then, the restriction of $\mathrm{cl}(\mathbb{R})$ to X becomes an isomorphism of the group X with the neutral component of $\mathrm{Pic}_C^0(\mathbb{R})$. In fact, the restriction of $\mathrm{cl}(\mathbb{R})$ to X is an isomorphism of Nash groups. The main ingredient of the proof is a result that establishes a large class of nonspecial divisors on C of relatively small degree.

In the paper, we will not discuss equations for the curve $C \subseteq \mathbb{P}^{2g}$. It is rather clear that there are 'generalized Weierstrass' equations for the curve $C \subseteq \mathbb{P}^{2g}$. Once one has such equations it should not be too difficult to determine explicitly the group law on the connected component X of $\mathrm{Div}_C^{g,\mathrm{eff}}(\mathbb{R})$.

Finally, we remark that all results of the paper hold for any real closed field instead of the ordinary field of real numbers. In particular, readers interested in algebraic curves defined over real number fields may think of $\mathbb{R}$ as the field $\mathbb{R}_{\mathrm{alg}}$ of real algebraic numbers.

Convention and notation. A real algebraic curve is a geometrically integral nonsingular proper scheme over $\mathbb{R}$ of dimension 1. The r-dimensional real projective space is simply denoted by $\mathbb{P}^r$ instead of $\mathbb{P}^r_{\mathbb{R}}$.

In this section we introduce some notation and terminology and briefly recall a result on nonspecial divisors on real algebraic curves [7]. For the convenience of the reader, we include its short proof.

Let C be a real algebraic curve and let D be a divisor on C. Let X be a connected component of the set of real points $C(\mathbb{R})$ of C and let $\mathrm{res}_X: \mathrm{Div}(C) \to \mathrm{Div}(C)$ be the restriction-to-X morphism. This morphism is defined by letting $\mathrm{res}_X(P) = P$ if $P \in X$ and $\mathrm{res}_X(P) = 0$ if $P \notin X$, for any closed point P of C. Now, for any divisor D on C, we define the degree of D on X to be the natural number $\deg_X(D) = \deg(\mathrm{res}_X(D))$.

Recall the following statement, of which we give a topological proof (cf. [2], Corollary 4.2.2).

Proposition 2.1. *Let C be a real algebraic curve. If ω is a nonzero rational differential form on C then $\deg_X(\mathrm{div}(\omega))$ is even for each connected component X of $C(\mathbb{R})$.*

Proof. The restriction of ω to X defines a nonzero meromorphic real differential form on the real analytic curve X. Since X is isomorphic to the real analytic curve $\mathbb{P}^1(\mathbb{R})$, the statement follows from the next lemma.

Lemma 2.2. *Let ω be a nonzero meromorphic real differential form on the real analytic curve $\mathbb{P}^1(\mathbb{R})$. Then, $\deg(\mathrm{div}(\omega))$ is even.*

Proof. Let η be the differential form $dx/(x^2+1)$ on $\mathbb{P}^1(\mathbb{R})$. Clearly, η has no zeros and no poles on $\mathbb{P}^1(\mathbb{R})$. Let f be the unique meromorphic real analytic function on $\mathbb{P}^1(\mathbb{R})$ such that $\omega = f \cdot \eta$. Then, $\mathrm{div}(\omega) = \mathrm{div}(f)$. If f is constant, $\mathrm{div}(f) = 0$. Hence, $\deg(\mathrm{div}(\omega))$ is even if f is constant. Suppose, therefore, that f is nonconstant and consider f as a real analytic map from $\mathbb{P}^1(\mathbb{R})$ into itself. Since f is nonconstant, $\mathrm{div}(f) = f^\star 0 - f^\star\infty$. But, it is easily seen that $\deg(f^\star 0)$ and $\deg(f^\star\infty)$ are both congruent to the topological degree mod 2 of f (see [8] for the definition and properties of the topological degree mod 2). Hence, $\deg(\mathrm{div}(f)) = \deg(f^\star 0) - \deg(f^\star\infty)$ is even. This shows that $\deg(\mathrm{div}(\omega))$ is even. $\square$

Recall that a divisor D on an algebraic curve C is said to be *nonspecial* if $h^0(D) = \deg(D) - g + 1$, where g is the genus of C. By Riemann-Roch, D is nonspecial if and only if $h^0(\Omega(-D)) = 0$. Here, $\Omega(-D)$ is the sheaf on C whose nonzero sections over an open subset U are the nonzero rational differential forms ω on C satisfying $\mathrm{div}(\omega) \geq D$ on U. It follows that, D is nonspecial if $D \geq D'$ and D' is nonspecial.

It follows from Riemann-Roch that divisors of degree at least $2g-1$ are nonspecial. Using Proposition 2 one gets a large class of nonspecial divisors of relatively small degree, i.e., of degree less than $2g-1$, on any real algebraic curve having real points:

Theorem 2.3. *Let C be a real algebraic curve and let g be its genus. Let D be a divisor on C and let d be its degree. Let k be the number of connected components X of $C(\mathbb{R})$ such that $\deg_X(D)$ is odd. If $d+k \geq 2g-1$ then D is nonspecial.*

Proof. We show that $h^0(\Omega(-D)) = 0$. Let ω be a global section of $\Omega(-D)$. Suppose that ω is nonzero. Then, ω is a rational section of Ω such that $\operatorname{div}(\omega) \geq D$. By Proposition 2, $\deg_X(\operatorname{div}(\omega))$ is even for each connected component X of $C(\mathbb{R})$. In particular, $\deg_X(\operatorname{div}(\omega))$ is at least $\deg_X(D)+1$ for each of the k connected components X of $C(\mathbb{R})$ for which $\deg_X(D)$ is odd. It follows that $\deg(\operatorname{div}(\omega)) \geq d+k \geq 2g-1$. Contradiction since $\deg(\operatorname{div}(\omega)) = 2g-2$. $\square$

Let C be a real algebraic curve, g its genus and s the number of connected components of $C(\mathbb{R})$. Harnack's Inequality for real algebraic curves states that $s \leq g+1$. Klein showed that, for all integers s and g satisfying $s \leq g+1$, there is a real algebraic curve C of genus g such that $C(\mathbb{R})$ has s connected components. In fact, there are many such curves since the moduli space of real algebraic curves C of genus g such that $C(\mathbb{R})$ has s connected components is a connected semianalytic variety of dimension $3g-3$ if $g \geq 2$ [5,11]. If $s = g+1$, C is called an *M-curve.* If $s = g$, C is called an $(M-1)$-curve. Let us say that a real algebraic curve C of genus g has *many real components* if C is an M-curve or an $(M-1)$-curve, i.e., if the number s of connected components of $C(\mathbb{R})$ is at least g.

3. THE NEUTRAL COMPONENT OF THE JACOBIAN

Let C be a real algebraic curve, g its genus and s the number of connected components of $C(\mathbb{R})$. The *Picard group* $\operatorname{Pic}(C)$ is the group of isomorphism classes of invertible sheaves on C. The subgroup $\operatorname{Pic}^0(C)$ of $\operatorname{Pic}(C)$ is the group of isomorphism classes of invertible sheaves on C of degree 0. It is a, not necessarily connected, compact commutative real Lie group of dimension g whose group of connected components is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{s-1}$ if $s \geq 1$ (see [2,3]). In fact, the group $\operatorname{Pic}^0(C)$ comes along with a finer structure: that of a real algebraic group, i.e., $\operatorname{Pic}^0(C)$ is a group object in the category of real algebraic varieties in the sense of [1]. Indeed, let Pic^0_C be the Jacobian variety of C over $\mathbb{R}$. Then, $\operatorname{Pic}^0(C)$ is nothing but the set of real points $\operatorname{Pic}^0_C(\mathbb{R})$ of Pic^0_C. Therefore, $\operatorname{Pic}^0(C)$ comes equipped with the structure of a real algebraic group in the sense of [1].

It follows form the preceding observations that the neutral real component $\operatorname{Pic}^0(C)^0$ of the Jacobian of C is a compact connected commutative real Lie group of dimension g. From the theory of real Lie groups it follows that $\operatorname{Pic}^0(C)^0$ is isomorphic to the real Lie group $(S^1)^g$. Again, the group $\operatorname{Pic}^0(C)^0$ comes naturally equipped with a finer structure: that of a Nash group.

Much in the spirit of [10] and stronger a definition than the one in [1], we define a *Nash manifold* to be a connected component of the set of real points of a real algebraic variety. A *morphism* of Nash manifolds is a map $f: M \to N$ such that f is the restriction of a morphism $F: X \to Y$ of real algebraic varieties and M and N are connected components of $X(\mathbb{R})$ and $Y(\mathbb{R})$, respectively. A *Nash group* is then defined to be a group object in the category of Nash manifolds. It is a trivial fact that $\mathrm{Pic}^0(C)^0$ is a Nash group. Note that this structure on $\mathrm{Pic}^0(C)^0$ is indeed finer than the real Lie group structure since $\mathrm{Pic}^0(C)^0$ is not isomorphic, as a Nash group, to the Nash group $(S^1)^g$. The neutral real component $\mathrm{Pic}^0(C)^0$ of the Jacobian of C can be intrinsically described as a subgroup of $\mathrm{Pic}(C)$ in the following way if $C(\mathbb{R}) \neq \emptyset$.

Let $\mathcal{L}$ be an invertible sheaf on C. Let $\pi: L \Rightarrow C$ be the geometric line bundle associated to $\mathcal{L}$ [4]. Then, π induces a topological line bundle $\pi(\mathbb{R}): L(\mathbb{R}) \to (\mathbb{R})$ over the topological manifold $C(\mathbb{R})$. Let

$$w = w_1(L(\mathbb{R})) \in H^1(C(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$$

be the first Stiefel-Whitney class of this topological line bundle [9]. Then, $\mathcal{L} \in \mathrm{Pic}^0(C)^0$ if and only if $w = 0$ [2,3]. In practice, $\mathcal{L}$ is the line bundle $\mathcal{O}(D)$ associated to a divisor D on C. Then, $w = 0$ if and only if the degree of D on any connected component X of $C(\mathbb{R})$ is even.

Now, the object of this paper is to show that, if the real algebraic curve C has many real components, there is a geometric description of the group $\mathrm{Pic}^0(C)^0$, much similar to the geometric description of the group structure on the neutral real component of a real elliptic curve [12]. In fact, for $g = 1$, our description coincides with the latter.

Form now on, assume that C is a real algebraic curve of genus $g \geq 1$ having many real components. Let $X_1, \ldots, X_g$ be g distinct connected components of $C(\mathbb{R})$ and put $X = X_1 \times \cdots \times X_g$. Clearly, X is a Nash manifold. Choose real points $O_i \in X_i$, for $i = 1, \ldots, g$ and let D be the divisor $3\sum O_i$. By Riemann-Roch, the linear system $|D|$ defines an embedding of C into $\mathbb{P}^{2g}$. We identify C with its image in $\mathbb{P}^{2g}$.

Define a map

$$\mathrm{cl}: X \to \mathrm{Pic}^0(C)^0$$

by

$$\mathrm{cl}(P_1, \ldots, P_g) = \mathcal{O}\Big(\sum_{i=1}^{g} (P_i - O_i)\Big),$$

where $(P_1, \ldots, P_g) \in X$. This map is well defined by the above observation, since the degree of the divisor $\sum(P_i - O_i)$ is even on each connected component of $C(\mathbb{R})$. The map cl is clearly a morphism of Nash manifolds. We will show that cl is an isomorphism of Nash manifolds. In fact, in [6], we showed already that this map is an isomorphism of Nash manifolds if C is an M-curve. Moreover, we are going to construct geometrically two laws on X, a unary law

$\ominus$ and a binary law $\oplus$. These laws, then, will correspond, through the map cl, to the Nash group laws on $\mathrm{Pic}^0(C)^0$.

The unary law is defined as follows. Let $(P_1, \ldots, P_g)$ be an element of X. The divisor $D' = D - \sum O_i - \sum P_i$ is of degree g and has degree 1 on each of the connected components X_i. Since $g + g \geq 2g - 1$, Theorem 2 states that the divisor D' is nonspecial, i.e., $h^0(D') = \deg(D') - g + 1 = 1$. This means geometrically that there is a unique hyperplane H of $\mathbb{P}^{2g}$ passing through $O_1, P_1, \ldots, O_g, P_g$.

Since the degree of D on each of the connected components X_i is odd, the same holds for the divisor $H \cdot C$ on C. Indeed, the divisor $(H \cdot C) - D$ is trivial in $\mathrm{Pic}(C)$. In particular, $(H \cdot C) - D$ belongs to the neutral component $\mathrm{Pic}^0(C)^0$ of the Jacobian. By the above observation, the degree of $(H \cdot C) - D$ is even on each connected component of $C(\mathbb{R})$. It follows that the degree of $H \cdot C$ on each of the connected components of X_i is odd. Hence, there are real points $Q_i \in X_i$ for $i = 1, \ldots, g$ such that $H \cdot C \geq \sum O_i + P_i + Q_i$. Since $H \cdot C$ is effective and of the same degree as D, i.e., of degree $3g$, one has

$$H \cdot C = \sum O_i + P_i + Q_i.$$

In particular, the real points $Q_i \in X_i$ are uniquely determined by the real points $P_i \in X_i$. Therefore, one can define a unary law $\ominus : X \to X$ by

$$\ominus(P_1, \ldots, P_g) = (Q_1, \ldots, Q_g).$$

One defines the binary law $\oplus$ on X as follows. Let $(P_1, \ldots, P_g)$ and $(Q_1, \ldots, Q_g)$ be in X. By Theorem 2, there is a unique hyperplane H of $\mathbb{P}^{2g}$ passing through $P_1, Q_1, \ldots, P_g, Q_g$. By the same argument as above, there are unique points $R_i \in X_i$, for $i = 1, \ldots, g$, such that

$$H \cdot C = \sum_{i=1}^{g} P_i + Q_i + R_i.$$

Define a binary law $\oplus : X \times X \to X$ by

$$(P_1, \ldots, P_g) \oplus (Q_1, \ldots, Q_g) = \ominus(R_1, \ldots, R_g).$$

One then has the following statement:

Theorem 3.1. *Let C be a real algebraic curve having many real components. Then, with notation as above, the map* $\mathrm{cl} : X \to \mathrm{Pic}^0(C)^0$ *is a Nash isomorphism of X onto* $\mathrm{Pic}^0(C)^0$*, considered as Nash manifolds. Moreover,*

$$\mathrm{cl}(P \oplus Q) = \mathrm{cl}(P) + \mathrm{cl}(Q) \quad \text{and} \quad \mathrm{cl}(\ominus P) = -\mathrm{cl}(P)$$

for all $P, Q \in X$. In particular, $(X, \oplus, \ominus)$ is a Nash group isomorphic to the Nash group $\mathrm{Pic}^0(C)^0$.

Proof. Let $\mathrm{Div}_C^{g,\mathrm{eff}}$ be the algebraic variety over $\mathbb{R}$ of effective divisors on C of degree g. In particular, $\mathrm{Div}_C^{g,\mathrm{eff}}(\mathbb{R})$ is the set of effective divisors on C of degree g. One can identify X with a connected component of $\mathrm{Div}_C^{g,\mathrm{eff}}(\mathbb{R})$

through the map $(P_1, \ldots, P_g) \mapsto \sum P_i$. The map $\mathrm{cl}\colon X \to \mathrm{Pic}^0(C)^0$ extends uniquely to a morphism of schemes over $\mathbb{R}$, again denoted by cl,

$$\mathrm{cl}\colon \mathrm{Div}_C^{g,\mathrm{eff}} \to \mathrm{Pic}_C^0.$$

The latter morphism is known to be birational. In fact, letting $U \subseteq \mathrm{Div}_C^{g,\mathrm{eff}}$ be the open subset of nonspecial divisors, $\mathrm{cl}(U)$ is open in Pic_C^0 and cl maps U isomorphically onto $\mathrm{cl}(U)$. By Theorem 2, the effective divisors corresponding to the elements of X are all nonspecial, i.e., $X \subseteq U$. Hence, X is a connected component of $U(\mathbb{R})$. Then, $\mathrm{cl}(X)$ is a connected component of $U(\mathbb{R})$. Since $\mathrm{cl}(X) \subseteq \mathrm{Pic}^0(C)^0$ and X is compact, $\mathrm{cl}(X) = \mathrm{Pic}^0(C)^0$. Therefore, the restriction of cl to X is a Nash isomorphism onto $\mathrm{Pic}^0(C)^0$.

Let $P, Q \in X$ and let $R \in X$ be such that $\ominus R = P \oplus Q$. It is clear from the construction of $\ominus P$ and R that $\mathrm{cl}(P) + \mathrm{cl}(\ominus P) = 0$ and that $\mathrm{cl}(P) + \mathrm{cl}(Q) + \mathrm{cl}(R) = 0$ in $\mathrm{Pic}(C)$. Hence, $\mathrm{cl}(\ominus P) = -\mathrm{cl}(P)$ and $\mathrm{cl}(P \oplus Q) = \mathrm{cl}(P) + \mathrm{cl}(Q)$. Therefore, $(X, \oplus, \ominus)$ is a Nash group and the map $\mathrm{cl}\colon X \to \mathrm{Pic}^0(C)^0$ is an isomorphism of Nash groups. □

As for real elliptic curves, one can deduce from the group structure on X the following geometric properties of the curve $C \subseteq \mathbb{P}^{2g}$:

Corollary 3.2. *There are exactly 2^g hyperplanes H in $\mathbb{P}^{2g}$ passing through the points $O_1, \ldots, O_g$ and tangent to each of the connected components $X_1, \ldots, X_g$ of $C(\mathbb{R})$.*

Corollary 3.3. *There are exactly 3^g hyperplanes H in $\mathbb{P}^{2g}$ such that the divisor $H \cdot C$ is divisible by 3.*

Corollary 3.4. *Let $(P_1, \ldots, P_g), (Q_1, \ldots, Q_g), (R_1, \ldots, R_g) \in X$ be such that $\sum P_i + Q_i + R_i$ is cut out by a hyperplane in $\mathbb{P}^{2g}$. If $3 \sum P_i$ and $3 \sum Q_i$ are cut out by hyperplanes then $3 \sum R_i$ is also cut out by a hyperplane in $\mathbb{P}^{2g}$.*

One can characterize the real algebraic curve $C \subseteq \mathbb{P}^{2g}$ by purely topological conditions and, then, get the above geometric properties as consequences, i.e.

Theorem 3.5. *Let $g \geq 1$ be an integer and let $C \subseteq \mathbb{P}^{2g}$ be a nondegenerate real algebraic curve of genus g and degree $3g$. Suppose that C has many real components and suppose that at least g of them represent the nontrivial homology class in $H_1(\mathbb{P}^{2g}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Then the following statements hold:*

1. There are exactly g connected components $X_1, \ldots, X_g$ of $C(\mathbb{R})$ representing the nontrivial homology class in $H_1(\mathbb{P}^{2g}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$.

2. There are real points $O_i \in X_i$, for $i = 1, \ldots, g$, such that the inclusion morphism of C into $\mathbb{P}^{2g}$ is the morphism associated to the linear system $|D|$, where $D = 3 \sum O_i$.

In particular, one can define, as above, a binary law $\oplus$ and a unary law $\ominus$ on $X = X_1 \times \cdots \times X_g$ such that X gets the structure of a Nash group.

Proof. Let $X_1, \ldots, X_g$ be connected components of $C(\mathbb{R})$ representing the nontrivial homology class in $H_1(\mathbb{P}^{2g}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Choose $P_i, Q_i \in X_i$, for $i = 1, \ldots, g$. Let $H \subseteq \mathbb{P}^{2g}$ be a hyperplane passing through $P_1, Q_1, \ldots, P_g, Q_g$. Since the degree of the divisor $H \cdot C$ on C is odd on each of the connected components $X_1, \ldots, X_g$ and since $H \cdot C$ is effective of degree $3g$, there are unique points $R_i \in X_i$ such that

$$H \cdot C = \sum P_i + Q_i + R_i.$$

Now, if there is a connected component X_0 of $C(\mathbb{R})$ different from $X_1, \ldots, X_g$ then the degree of $H \cdot C$ on X_0 is equal to 0. In particular, the degree of $H \cdot C$ on X_0 is even. Hence, the connected component X_0 of $C(\mathbb{R})$ is homologous to 0 in $H_1(\mathbb{P}^{2g}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. This shows Statement 1.

Let $f\colon C \to \mathbb{P}^{2g}$ be the inclusion morphism and let $H \subseteq \mathbb{P}^{2g}$ be a hyperplane. Put $D' = f^\star H$. Then, D' is a divisor on C of degree $3g$. Since $3g \geq 2g - 1$, D' is nonspecial. Hence, $h^0(D') = 3g - g + 1 = 2g + 1$ and the linear system $|D'|$ is of dimension $2g$. This shows that f is the morphism associated to D'. In order to finish the proof of Statement 2, we have to show that there are $O_i \in X_i$, for $i = 1, \ldots, g$ such that D' is equivalent to D, where $D = 3 \sum O_i$.

If C is an M-curve, denote by X_0 the unique connected component of $C(\mathbb{R})$ homologous to 0 in $H_1(\mathbb{P}^{2g}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Otherwise, let X_0 denote the empty set. Let $\mathcal{P}$ be the connected component of $\mathrm{Pic}^g(C)$ of isomorphism classes of line bundles $\mathcal{L}$ of degree g on C such that the first Stiefel-Whitney class of the restriction of $\mathcal{L}$ to X_i is nonzero, for $i = 1, \ldots, g$, and is equal to 0 for $i = 0$. Define similarly a connected component $\mathcal{P}'$ of $\mathrm{Pic}^{3g}(C)$. One has $\mathcal{O}(D') \in \mathcal{P}'$ and $3\mathcal{P} = \mathcal{P}'$. By Theorem 3, any line bundle $\mathcal{L}$ in $\mathcal{P}$ is isomorphic to a line bundle of the form $\mathcal{O}(D'')$, where $D'' = \sum O_i$ with $O_i \in X_i$ for $i = 1, \ldots, g$. Therefore, Statement 2 follows. $\square$

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